Lecture 4.

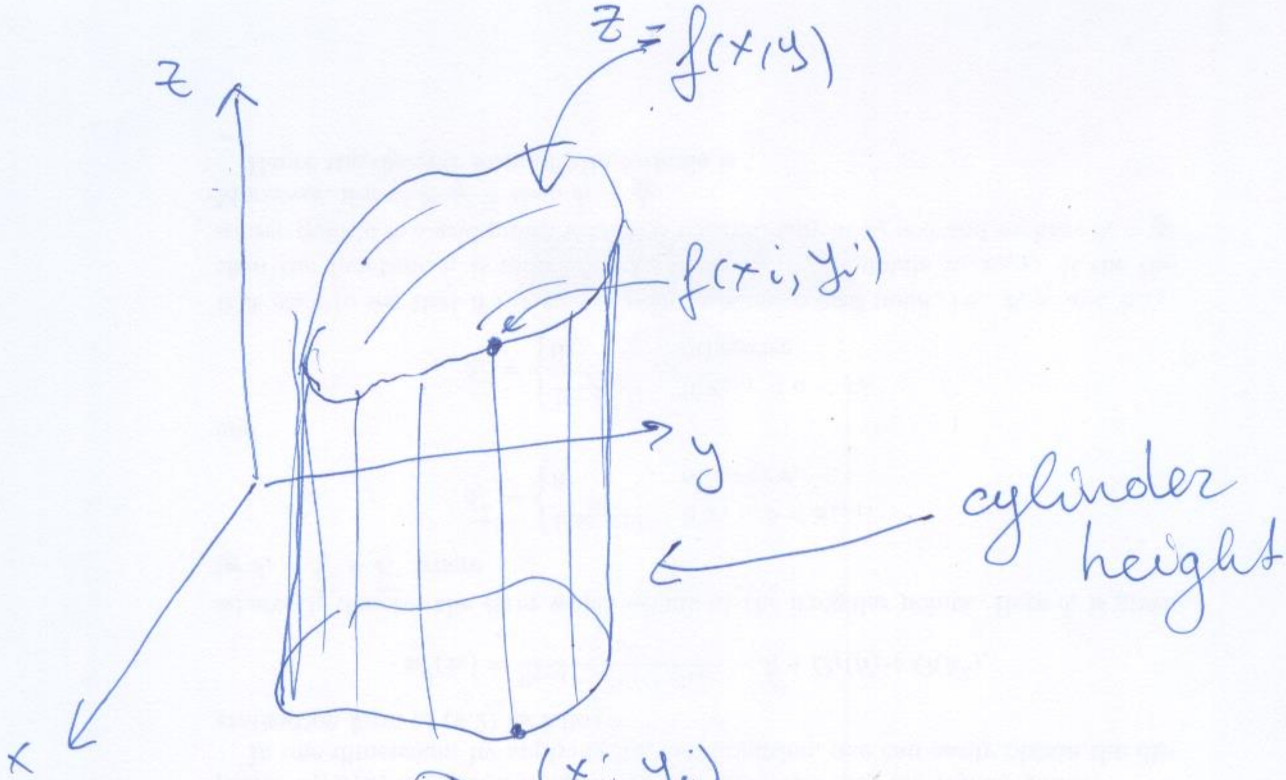
Double integrals

A double integral is a definite integral of a function of two variables over a region D in R^2.

Let's assume that $f(x,y)$ is defined in a domain $D \subset R^2$.

D is bounded and its boundary is defined by smooth curves

As an important application we consider the task to find the volume V of the cylinder which is bounded by xOy plane from below, by the surface $f(x,y)$ from above, and cylindrical surface defined on domain D.



1. Consider a partition of D into a finite number of non-overlapping sub-regions

$$D = \bigcup_{i=1}^{n} D_i = D_1 \cup D_2 \cup \dots \cup D_n$$

2. We denote by ΔS_i the ~~array~~ area of D_i, $i = 1, \dots, n$.

d_i is the diameter of D_i,

$d = \max\limits_{1 \le i \le n} d_i$, the largest diameter

3. P_i is a point in D_i.

$P_i = (x_i, y_i)$, $i = 1, \dots, n$.

4. Calculate the Riemann sum

$$\sigma_n = \sum_{i=1}^{n} f(x_i, y_i)\,\Delta S_i$$

Definition The function f is said to be Riemann integrable in region D if the limit

$$V = \lim_{d \to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta S_i$$

exists and don't depend on a selection of a partition of D and set of points P_i, $i = 1, \dots, 2$.

This integral is denoted

$$V = \iint_D f(x,y)\,dx\,dy \quad \text{or} \quad \iint_D f(x,y)\,dS.$$

Sufficient and necessary conditions for the existance of the limit can be proved as in the case of one variable functions.

Darboux sums

Let

$$M_i = \sup_{P \in D_i} f(P) \quad (\text{the supremum})$$

$$m_i = \inf_{P \in D_i} f(P) \quad (\text{the infimum}).$$

The upper Darboux sum with respect to partition D_i, $i = 1, \dots, n$

$$U_n = \sum_{i=1}^{n} M_i \, \Delta S_i$$

The lower Darboux sum with respect to partition D_i, $i=1,\dots,n$

$$L_n = \sum_{i=1}^{n} m_i \,\Delta S_i.$$

Theorem 1 For any partitions (different for U_n, L_m) we have estimates:

$$L_m \le U_n.$$

Theorem 2 Function $f(x,y)$ is integrable in region D if and only if (iff) for any $\varepsilon > 0$ there exists a partition such that

$$\sum_{i=1}^{n} \omega_i \,\Delta S_i < \varepsilon \quad \left(\text{or } U_n - L_n < \varepsilon\right),$$

where $\omega_i = M_i - m_i$, $i = 1, 2, \dots, n$.

Theorem 3. Any continuous function $f(x,y)$ is Riemann integrable in a bounded domain D.

In this case f is uniformly continuous.

Theorem 4. If both functions f and g are integrable, then

$$\iint_D \left(\alpha f(x,y) + \beta g(x,y)\right) dx\,dy = \alpha \iint_D f(x,y)\,dx\,dy + \beta \iint_D g(x,y)\,dx\,dy.$$

Theorem 5. If $f \geq 0$ for $(x,y) \in D$, then

$$\iint_D f(x,y)\,dS \geq 0.$$

Theorem 6. If $f \geq g$, $(x,y) \in D$,

then

$$\iint_D f(x,y)\,dS \geq \iint_D g(x,y)\,dS.$$

Theorem 7. If $D = D_1 \cup D_2$ and D_1 and D_2 are not-overlapping, then

$$\iint_D f(x,y)\,dS = \iint_{D_1} f(x,y)\,dS + \iint_{D_2} f(x,y)\,dS.$$

Proof It was proved that the limit, if it exist, don't depend on a partition. Then we can take sets of partitions when each sub-region depends only to one ~~or~~ region D_1 and D_2. ▷

Let denote

$$m = \min_{P \in D} f(x,y), \quad M = \max_{P \in D} f(x,y).$$

Then

$$m\, S \le \iint_D f(x,y)\,dx\,dy \le M\, S,$$

where S is the area of region D.

Since $f(x,y)$ is a continuous function in D, then $\exists(\bar{x},\bar{y}) \in D$

$$\iint_D f(x,y)\,dx\,dy = f(\bar{x},\bar{y})\, S.$$

$f(\bar{x},\bar{y})$ is called the mean value of $f(x,y)$ in region D.